Published by Institute of Physics Publishing for SISSA

Received: *April 14, 2006*
Revised: *December 19, 2006*
Accepted: *January 12, 2007*
Published: *February 12, 2007*

The index of the overlap Dirac operator on a discretized 2d non-commutative torus

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Abstract: The index, which is given in terms of the number of zero modes of the Dirac operator with definite chirality, plays a central role in various topological aspects of gauge theories. We investigate its properties in non-commutative geometry. As a simple example, we consider the U(1) gauge theory on a discretized 2d non-commutative torus, in which general classical solutions are known. For such backgrounds we calculate the index of the overlap Dirac operator satisfying the Ginsparg-Wilson relation. When the action is small, the topological charge defined by a naive discretization takes approximately integer values, and it agrees with the index as suggested by the index theorem. Under the same condition, the value of the index turns out to be a multiple of N, the size of the 2d lattice. By interpolating the classical solutions, we construct explicit configurations, for which the index is of order 1, but the action becomes of order N. Our results suggest that the probability of obtaining a non-zero index vanishes in the continuum limit, unlike the corresponding results in the commutative space.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry, Lattice Gauge Field Theories.

Contents

1. Introduction

Non-commutative (NC) geometry [1, 2] has been studied for quite a long time as a simple modification of our notion of space-time at small distances possibly due to effects of quantum gravity [3]. It has attracted much attention since it was shown to appear naturally from matrix models [4, 5] and string theories [6]. In particular, field theory on NC geometry has a peculiar property known as the UV/IR mixing [7], which may cause a drastic change of the long-distance physics through quantum effects. This phenomenon has been first discovered in perturbation theory, but it was shown to appear also in a fully nonperturbative setup [8]. A typical example is the spontaneous breaking of the translational symmetry in NC scalar field theory, which was first conjectured from a self-consistent one-loop analysis [9] and confirmed later on by Monte Carlo simulation [10 − 12]. (See also [13, 14].)

The appearance of a new type of IR divergence due to the UV/IR mixing spoils the perturbative renormalizability in general [15], and therefore, even the existence of a sensible field theory on a NC geometry is *a priori* debatable. In order to study such a nonperturbative issue, one has to define a regularized field theory on NC geometry, which is possible by using matrix models. In the case of NC torus, for instance, the so-called twisted reduced model [16, 17] is interpreted as a lattice formulation of NC field theories [8], in which finite N matrices are mapped one-to-one onto fields on a periodic lattice. The existence of a sensible continuum limit and hence the nonperturbative renormalizability have been shown by Monte Carlo simulations in NC U(1) gauge theory in 2d [18] and 4d [19] as well as in NC scalar field theory in 3d [12, 20].

In the case of fuzzy sphere [21], finite N matrices are mapped one-to-one onto functions on the sphere with a specific cutoff on the angular momentum. The fuzzy sphere (or fuzzy manifolds [22, 23] in general) preserves the continuous symmetry of the base manifold, which makes it an interesting candidate for a novel regularization of *commutative* field theories alternative to the lattice [24]. It is also interesting to use fuzzy spheres in the coset space dimensional reduction [25]. Stability of fuzzy manifolds in matrix models with the Chern-Simons term [26, 27] has been studied by Monte Carlo simulations [28, 29].

One of the interesting features of NC field theories is the appearance of a new type of topological objects, which are referred to as NC solitons [30], NC monopoles, NC instantons, and fluxons [31] in the literature. They are constructed by using a projection operator, and the matrices describing such configurations are assumed to be infinite dimensional. In finite NC geometries, namely in the case where field configurations are described by finite-dimensional matrices and therefore regularized, topological objects have been constructed by using the algebraic K-theory and projective modules [32 – 34].

Dynamical aspects of these topological objects are important in particular in the realization of a 4d chiral gauge theory in the context of string theory compactification, which requires a nontrivial index in the compactified dimensions (See, for instance, section 14 of ref. [35].) Ultimately we hope to realize such a scenario *dynamically*, for instance, in the IIB matrix model [36], in which the dynamical generation of *four-dimensional* space-time [37 – 39] as well as the gauge group [40, 41] has been studied intensively.

Extending the notion of the index to *finite* NC geometry is a nontrivial issue due to the doubling problem of the naive Dirac action. The same problem occurs also in the ordinary lattice gauge theory. There one can add the Wilson term to the Dirac action to remove the species doublers in the continuum limit at the expense of the explicit breaking of chiral symmetry. This had been a notorious problem in lattice gauge theory as manifested by the no-go theorem [42]. It was found, however, that by adopting a Dirac operator which satisfies the Ginsparg-Wilson relation [43], a modified chiral symmetry, which becomes the usual one in the continuum limit, can be exactly preserved on the lattice [44, 45]. The Dirac operator has exact zero modes with definite chirality for topologically nontrivial gauge configurations, and therefore one can define the index unambiguously [46, 47, 44]. A concrete example of such an operator with desirable properties in the continuum limit is given by the so-called overlap[1] Dirac operator [48]. The index theorem for the overlap Dirac operator is studied numerically in refs. [46, 49]. The successful results obtained in these tests may be understood from the analytical work [50] (See also ref. [51] for an extension.), in which the usual expression for the topological charge in the continuum has been derived from the index of the overlap Dirac operator *nonperturbatively*. As emphasized

[1]Historically, the overlap formalism [46], from which one can actually derive the overlap Dirac operator [48], has been established before the rediscovery of the Ginsparg-Wilson relation.

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in refs. [50], the derivation of the correct axial anomaly [52], which uses the perturbative expansion with respect to the gauge field, is not sufficient for demonstrating the index theorem for topologically nontrivial gauge configurations.

In the past several years, the ideas developed in lattice gauge theory have been successfully extended to NC geometry. In the case of NC torus, the overlap Dirac operator has been introduced in ref. [53], and it was used to define a NC chiral gauge theory with manifest star-gauge invariance. For general NC manifolds, a prescription to define the Ginsparg-Wilson Dirac operator and its index has been provided in ref. [54], and the fuzzy sphere was considered as a concrete example.[2] The Ginsparg-Wilson algebra for the fuzzy sphere has been studied in detail in each topological sector [33]. In ref. [56] the overlap Dirac operator on the NC torus [53] was derived also from this general prescription [54], and the axial anomaly has been calculated in the continuum limit.

In an attempt to construct a topologically nontrivial configuration on the fuzzy sphere, an analogue of the 't Hooft-Polyakov monopole was obtained [33, 34]. Although the index defined through the Ginsparg-Wilson Dirac operator vanishes for these configurations, one can make it non-zero by inserting a projection operator, which picks up the unbroken U(1) component of the SU(2) gauge group. In fact the 't Hooft-Polyakov monopole configurations are precisely the meta-stable states observed in Monte Carlo simulations [28] taking the two coincident fuzzy spheres as the initial configuration, which eventually decays into a single fuzzy sphere. In ref. [57] this instability was studied analytically by the one-loop calculation of free energy around the 't Hooft-Polyakov monopole configurations, and it was interpreted as the dynamical generation of a nontrivial index, which may be used for the realization of a chiral fermion in our space-time.

The primary aim of the present work is to investigate the properties of the index in finite NC geometry, taking the 2d U(1) gauge theory on a discretized NC torus as a simple example, which is studied extensively in the literature both numerically [18] and analytically [58 – 60]. In particular, ref. [60] presents general classical solutions carrying the topological charge. We compute the index defined through the overlap Dirac operator for these classical solutions. The topological charge defined naively on the discretized NC torus is not an integer in general, although the index is. We observe, however, that when the action is small, the topological charge is close to an integer, and it agrees with the index as suggested by the index theorem. In fact, under the same condition, the index turns out to be a multiple of N, the linear size of the 2d lattice. By interpolating the classical solutions, we construct explicit configurations[3] for which the index is of order 1, but the action becomes of order N. Our results suggest that the probability of obtaining a non-zero index vanishes in the continuum limit, which is consistent with the instanton calculus in the continuum theory [59].

The rest of this paper is organized as follows. In section 2 we provide some generalities concerning a matrix model formulation of gauge theories on a discretized NC torus and define the index of the overlap Dirac operator. In section 3 we focus on the two-dimensional

[2] The Ginsparg-Wilson Dirac operator for vanishing gauge field was constructed earlier in refs. [55].

[3] While we were preparing this article, we received a preprint [61], in which a gauge configuration with non-zero index was found numerically in the same model at small N.

case, and discuss the classical solutions and the topological charge. In section 4 we examine whether the index theorem holds for the classical solutions. In section 5 we construct explicit configurations with the index of order 1 by interpolating the classical solutions, and study their properties. In section 6 we review some known results in the commutative case, and discuss their relationship to our results. Section 7 is devoted to a summary and discussions.

2. Generalities

2.1 Gauge theory on a discretized NC torus

The U(1) gauge theory on a NC space is given by the action

$$S_{\rm cont} = \frac{1}{g^2}\int \mathrm{d}^d x \; \frac{1}{4}\Big(F_{\mu\nu}(x)\star F_{\mu\nu}(x)\Big) \, , \tag{2.1}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i\,\{A_\mu(x)\star A_\nu(x) - A_\nu(x)\star A_\mu(x)\} \, , \tag{2.2}$$

where the star-product is defined by

$$\varphi_1(x)\star\varphi_2(x) = \exp\left(\frac{i}{2}\Theta_{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\mu}\right)\varphi_1(x)\varphi_2(y)\bigg|_{x=y} \; . \tag{2.3}$$

Note that the star-product is associative but non-commutative. This non-commutativity may be attributed to that of space-time since

$$x_\mu\star x_\nu - x_\nu\star x_\mu = i\Theta_{\mu\nu} \; . \tag{2.4}$$

The action (2.1) is invariant under a star-gauge transformation

$$A_\mu(x) \mapsto g(x)\star A_\mu(x)\star g^*(x) - i g(x)\star\partial_\mu g^*(x) \, , \tag{2.5}$$

where $g(x)$ obeys the star-unitarity condition

$$g(x)\star g(x)^* = g(x)^*\star g(x) = 1 \tag{2.6}$$

instead of $|g(x)| = 1$.

When we discretize the space, the consistency with the NC algebra inevitably requires the space to be compactified in a specific way [8]. Thus we obtain a theory on a periodic L^d lattice with the action

$$S_{\rm lat} = -\beta\sum_x\sum_{\mu\neq\nu} U_\mu(x)\star U_\nu(x+a\hat\mu)\star U_\mu(x+a\hat\nu)^*\star U_\nu(x)^* \, , \tag{2.7}$$

where the link variables $U_\mu(x)$ are star-unitary; i.e.,

$$U_\mu(x)\star U_\mu(x)^* = U_\mu(x)^*\star U_\mu(x) = 1 \; . \tag{2.8}$$

We use the standard notation in lattice gauge theory, where $\hat\mu$ represents a unit vector in the μ direction, and a represents the lattice spacing. The star-product on the lattice can be obtained by rewriting (2.3) in terms of Fourier modes and restricting the momentum to be the one allowed on the lattice. As in the commutative space, one obtains the continuum action (2.1) from (2.7) in the $a \to 0$ limit with the identification $\beta = \frac{1}{2a^2 g^2}$ and

$$U_\mu(x) = \mathcal{P} \exp_\star \left(i \int_x^{x+a\hat{\mu}} dz A_\mu(z) \right) . \tag{2.9}$$

2.2 The overlap Dirac operator and its index

In this section we define the overlap Dirac operator and its index for a gauge configuration on the discretized NC torus [53, 54, 56]. All the formulae have the same form as in the usual lattice gauge theory except for the use of the star product.

We consider a Dirac operator D satisfying the Ginsparg-Wilson relation [43]

$$\gamma_5 D + D\gamma_5 = a\, D\gamma_5 D\ . \tag{2.10}$$

Assuming the γ_5-hermiticity $D^\dagger = \gamma_5 D \gamma_5$, we can define a hermitian operator $\hat{\gamma}_5$ by

$$\hat{\gamma}_5 = \gamma_5 \left(1 - aD\right) , \tag{2.11}$$

which may be solved for D as $D = \frac{1}{a}(1 - \gamma_5\hat{\gamma}_5)$. Then the Ginsparg-Wilson relation (2.10) is equivalent to requiring $\hat{\gamma}_5$ to be unitary. The overlap Dirac operator corresponds to taking $\hat{\gamma}_5$ to be [48]

$$\hat{\gamma}_5 = \frac{H}{\sqrt{H^2}} , \tag{2.12}$$

$$H = \gamma_5 \left(1 - aD_{\mathrm{W}}\right) , \tag{2.13}$$

where D_{W} is the Wilson-Dirac operator

$$D_{\mathrm{W}} = \frac{1}{2} \sum_{\mu=1}^{d} \left\{ \gamma_\mu \left(\nabla_\mu^* + \nabla_\mu \right) - a \nabla_\mu^* \nabla_\mu \right\} \tag{2.14}$$

with ∇_μ (∇_μ^*) being the covariant forward (backward) difference operator defined by

$$\nabla_\mu \Psi(x) = \frac{1}{a} \left[U_\mu(x) \star \Psi(x + a\hat{\mu}) - \Psi(x) \right] , \tag{2.15}$$

$$\nabla_\mu^* \Psi(x) = \frac{1}{a} \left[\Psi(x) - U_\mu(x - a\hat{\mu})^\dagger \star \Psi(x - a\hat{\mu}) \right] . \tag{2.16}$$

Since the Ginsparg-Wilson relation (2.10) can be rewritten as

$$\gamma_5 D + D\hat{\gamma}_5 = 0 , \tag{2.17}$$

the lattice action

$$S = a^d \sum_x \bar{\Psi}(x) \star D \Psi(x) \tag{2.18}$$

has the exact lattice chiral symmetry [44, 45]

$$\Psi(x) \mapsto \mathrm{e}^{i\alpha\hat{\gamma}_5}\, \Psi(x) , \qquad \bar{\Psi}(x) \mapsto \bar{\Psi}(x)\, \mathrm{e}^{i\alpha\gamma_5} . \tag{2.19}$$

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Note also that the space of zero modes of the Dirac operator D is invariant under γ_5, which means that one can define the index of D unambiguously by $\nu \equiv n_+ - n_-$, where $n_\pm$ is the number of zero modes with the chirality ± 1. It turns out that [46, 47, 44]

$$\nu = \frac{1}{2}\mathcal{T}r(\gamma_5 + \hat{\gamma}_5) = \frac{1}{2}\mathcal{T}r\frac{H}{\sqrt{H^2}}\,, \tag{2.20}$$

where $\mathcal{T}r$ represents a trace over the space of the Dirac spinor field on the lattice.

2.3 Matrix formulation

So far we have been using a formulation of NC geometry, in which the non-commutativity of the space-time is encoded in the star-product. In fact it is much more convenient for our purpose to use an equivalent formulation [62], in which one maps functions on a NC space to operators so that the star-product becomes nothing but the usual operator product, which is non-commutative. In particular, the coordinate operators $\hat{x}_\mu$ satisfy the commutation relation $[\hat{x}_\mu, \hat{x}_\nu] = i\Theta_{\mu\nu}$. In the discrete version, one maps a field $\varphi(x)$ on the L^d lattice onto a $N \times N$ matrix Φ, where $N^2 = L^d$ in order to match the degrees of freedom. This map yields the following correspondence

$$\varphi_1(x) \star \varphi_2(x) \Leftrightarrow \Phi_1\Phi_2\,, \tag{2.21}$$

$$\varphi(x + a\hat{\mu}) \Leftrightarrow \Gamma_\mu \Phi \Gamma_\mu^\dagger\,, \tag{2.22}$$

$$\frac{1}{L^d}\sum_x \varphi(x) \Leftrightarrow \frac{1}{N}\mathrm{tr}\,\Phi\ . \tag{2.23}$$

The SU(N) matrices Γ_μ ($\mu = 1, \cdots, d$) represent a shift on the matrix side, and they satisfy the 't Hooft-Weyl algebra

$$\Gamma_\mu\Gamma_\nu = \mathcal{Z}_{\mu\nu}\Gamma_\nu\Gamma_\mu\,, \tag{2.24}$$

where $\mathcal{Z}_{\mu\nu} = \mathcal{Z}_{\nu\mu}^*$ is a phase factor. An explicit representation of Γ_μ in the $d = 2$ case shall be given in section 3.1.

Using the map, one can reformulate the lattice theory (2.7) in terms of matrices. The star-unitarity condition (2.8) on the link variables $U_\mu(x)$ simply implies that the corresponding matrix $\hat{U}_\mu$ should be unitary. The action (2.7) can be written as

$$\begin{aligned} S &= -N\beta\sum_{\mu\neq\nu}\mathrm{tr}\,\left\{\hat{U}_\mu\,(\Gamma_\mu\hat{U}_\nu\Gamma_\mu^\dagger)\,(\Gamma_\nu\hat{U}_\mu^\dagger\Gamma_\nu^\dagger)\,\hat{U}_\nu^\dagger\right\} + 2\beta N^2 && (2.25)\\ &= -N\beta\sum_{\mu\neq\nu}\mathcal{Z}_{\nu\mu}\mathrm{tr}\,\left(V_\mu\,V_\nu\,V_\mu^\dagger\,V_\nu^\dagger\right) + 2\beta N^2\,, && (2.26)\end{aligned}$$

where $V_\mu \equiv \hat{U}_\mu\Gamma_\mu$ is a U(N) matrix. This is nothing but the twisted Eguchi-Kawai (TEK) model [17], which appeared in history as a matrix model equivalent to the large N gauge theory [16]. In fact we have added the constant term $2\beta N^2$ to what we would obtain from (2.7) in order to make the absolute minimum of the action zero. We use this convention in the rest of this paper.

The index can be calculated using eq. (2.20), where H is defined by eq. (2.13). The only thing to note in transcription into matrices is that the covariant forward and backward difference operators ∇_μ and ∇_μ^*, which appear in the definition of the Wilson-Dirac operator (2.14), should now be defined as

$$\nabla_\mu \Psi = \frac{1}{a}\left[\hat{U}_\mu(\Gamma_\mu \Psi \Gamma_\mu^\dagger) - \Psi\right] = \frac{1}{a}\left[V_\mu \Psi \Gamma_\mu^\dagger - \Psi\right] , \tag{2.27}$$

$$\nabla_\mu^* \Psi = \frac{1}{a}\left[\Psi - (\Gamma_\mu^\dagger \hat{U}_\mu^\dagger \Gamma_\mu)(\Gamma_\mu^\dagger \Psi \Gamma_\mu)\right] = \frac{1}{a}\left[\Psi - V_\mu^\dagger \Psi \Gamma_\mu\right] . \tag{2.28}$$

The index is simply given by half the difference between the numbers of positive and negative eigenvalues of the hermitian matrix H. In this calculation we may simply set $a = 1$, since the lattice spacing a appearing in the definition of the index actually cancels out as it should. The computational effort for calculating the index is of order N^6, since we have to diagonalize the $2N^2 \times 2N^2$ hermitian matrix H.

3. Two-dimensional case

In this section we focus on the two-dimensional case, and discuss the classical solutions and the topological charge.

3.1 Explicit representation

An explicit form of the map between fields and matrices in the two-dimensional case is given, for instance, in ref. [53], where the twist in eq. (2.24) is given by

$$\mathcal{Z}_{12} = \exp\left(2\pi i \frac{M}{N}\right) , \qquad M = \frac{N+1}{2} \tag{3.1}$$

with N being an odd integer. The algebra (2.24) can be realized by

$$\Gamma_1 = \mathcal{P}_N , \quad \Gamma_2 = (\mathcal{Q}_N)^M , \tag{3.2}$$

where we have defined the SU(n) matrices

$$\mathcal{P}_n = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} , \qquad \mathcal{Q}_n = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i/n} & & & \\ & & e^{4\pi i/n} & & \\ & & & \ddots & \\ & & & & e^{2\pi i(n-1)/n} \end{pmatrix} \tag{3.3}$$

obeying $\mathcal{P}_n \mathcal{Q}_n = e^{2\pi i/n}\, \mathcal{Q}_n \mathcal{P}_n$ for later convenience. For this particular construction, which we are going to use throughout this paper, it turns out that the NC tensor, which appears in the star-product, is given by

$$\Theta_{\mu\nu} = \vartheta\, \epsilon_{\mu\nu} , \tag{3.4}$$

$$\vartheta = \frac{1}{\pi} N a^2 . \tag{3.5}$$

Note that the linear size of the torus $\ell = Na$ goes to ∞ in the continuum limit $a \to 0$ fixing ϑ. A finite torus can be obtained by other constructions given in the first paper of refs. [8] and ref. [60], which are mutually equivalent.

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3.2 Definition of the topological charge

Let us define the topological charge for a gauge configuration on the discretized 2d torus. In the language of fields, we define the topological charge as

$$Q = \frac{1}{4\pi i} \sum_x \sum_{\mu\nu} \epsilon_{\mu\nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^* \star U_\nu(x)^* \,, \tag{3.6}$$

which is obtained as a *naive* discretization of the topological charge in 2d gauge theory defined in the continuum as

$$Q = \frac{1}{4\pi} \int d^2x \, \epsilon_{\mu\nu} F_{\mu\nu} \ . \tag{3.7}$$

By using the map between fields and matrices, the topological charge (3.6) can be represented in terms of matrices as

$$\begin{aligned} Q &= \frac{1}{4\pi i} N \sum_{\mu\nu} \epsilon_{\mu\nu} \mathrm{tr} \, \left\{ \hat{U}_\mu \, (\Gamma_\mu \hat{U}_\nu \Gamma_\mu^\dagger) \, (\Gamma_\nu \hat{U}_\mu^\dagger \Gamma_\nu^\dagger) \, \hat{U}_\nu^\dagger \right\} & (3.8) \\ &= \frac{1}{4\pi i} N \sum_{\mu\nu} \epsilon_{\mu\nu} \mathcal{Z}_{\nu\mu} \mathrm{tr} \, \left(V_\mu \, V_\nu \, V_\mu^\dagger \, V_\nu^\dagger \right) \ . & (3.9) \end{aligned}$$

3.3 Classical solutions

The classical equation of motion can be obtained from the action (2.26) as

$$V_\mu^\dagger (W - W^\dagger) V_\mu = W - W^\dagger \,, \tag{3.10}$$

where the unitary matrix W is defined by

$$W = \mathcal{Z}_{\nu\mu} V_\mu \, V_\nu \, V_\mu^\dagger \, V_\nu^\dagger \ . \tag{3.11}$$

The general solutions[4] to this equation can be brought into a block-diagonal form [60]

$$V_\mu = \begin{pmatrix} \Gamma_\mu^{(1)} & & & \\ & \Gamma_\mu^{(2)} & & \\ & & \ddots & \\ & & & \Gamma_\mu^{(k)} \end{pmatrix} \tag{3.12}$$

by an appropriate $\mathrm{SU}(N)$ transformation, where $\Gamma_\mu^{(j)}$ are $n_j \times n_j$ unitary matrices satisfying the 't Hooft-Weyl algebra

$$\begin{aligned} \Gamma_\mu^{(j)} \Gamma_\nu^{(j)} &= Z_{\mu\nu}^{(j)} \Gamma_\nu^{(j)} \Gamma_\mu^{(j)} \ , & (3.13) \\ Z_{12}^{(j)} &= Z_{21}^{(j)*} = \exp\left(2\pi i \frac{m_j}{n_j} \right) \ . & (3.14) \end{aligned}$$

An explicit representation [63] is given, for instance, by

$$\Gamma_1^{(j)} = \mathcal{P}_{n_j} \,, \quad \Gamma_2^{(j)} = (\mathcal{Q}_{n_j})^{m_j} \ . \tag{3.15}$$

[4]In fact there is another type of solutions, which we do not consider in this paper since they do not have finite action in the continuum limit [60].

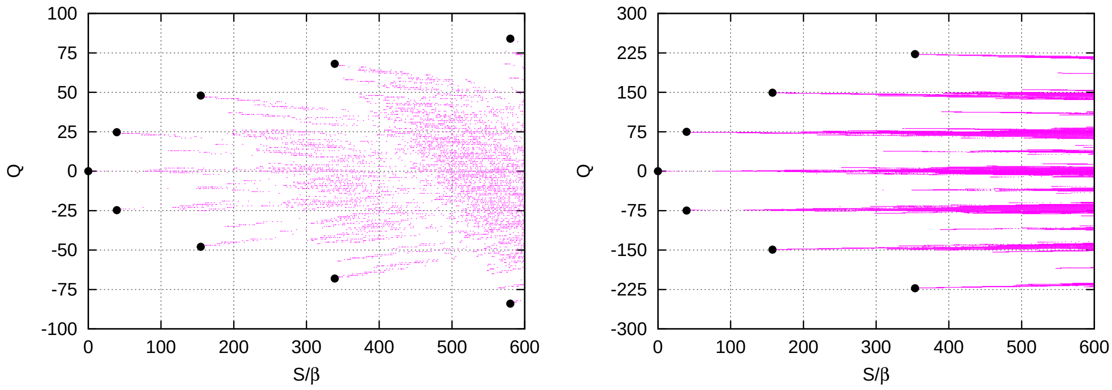

Figure 1: A scatter plot of the action S/β (x-axis) and the topological charge Q (y-axis) for classical solutions at $N = 25$ (left) and $N = 75$ (right). The closed circles represent the solutions for $k = 1$, which gives (4.1).

For each solution, the action and the topological charge can be easily evaluated as

$$S = 4N\beta \sum_j n_j \sin^2 \left\{ \pi \left(\frac{m_j}{n_j} - \frac{M}{N} \right) \right\} , \tag{3.16}$$

$$Q = \frac{N}{2\pi} \sum_j n_j \sin \left\{ 2\pi \left(\frac{m_j}{n_j} - \frac{M}{N} \right) \right\} . \tag{3.17}$$

Note that the topological charge Q is not an integer in general. If we require the action to be less than of order N, however, the argument of the sine has to vanish in the large N limit for all j . In that case the topological charge approaches an integer

$$Q \simeq N \left(\sum_j m_j - M \right) , \tag{3.18}$$

which is actually a multiple of N.

4. The index theorem for the classical solutions

The index theorem [64] relates the topological charge of an arbitrary gauge configuration to the index of the Dirac operator on that background. A proof of the index theorem in noncommutative $\mathbb{R}^d$ is given in ref. [65].[5] However, as in the commutative case, the formulation of the index theorem becomes nontrivial in the discretized setup. Here we address this issue for the classical solutions reviewed in the previous section.

Let us first look at the spectrum of the topological charge for the classical solutions. In figure 1 we present a scatter plot of the action S/β (x-axis) and the topological charge Q (y-axis) for the classical solutions at $N = 25$ (left) and $N = 75$ (right). We plot all the

[5]In ref. [65] an explicit ADHM construction of the fermionic zero modes in the multi-instanton backgrounds was also performed, and the number of zero modes agreed with the index theorem. See also ref. [66] for a related work.

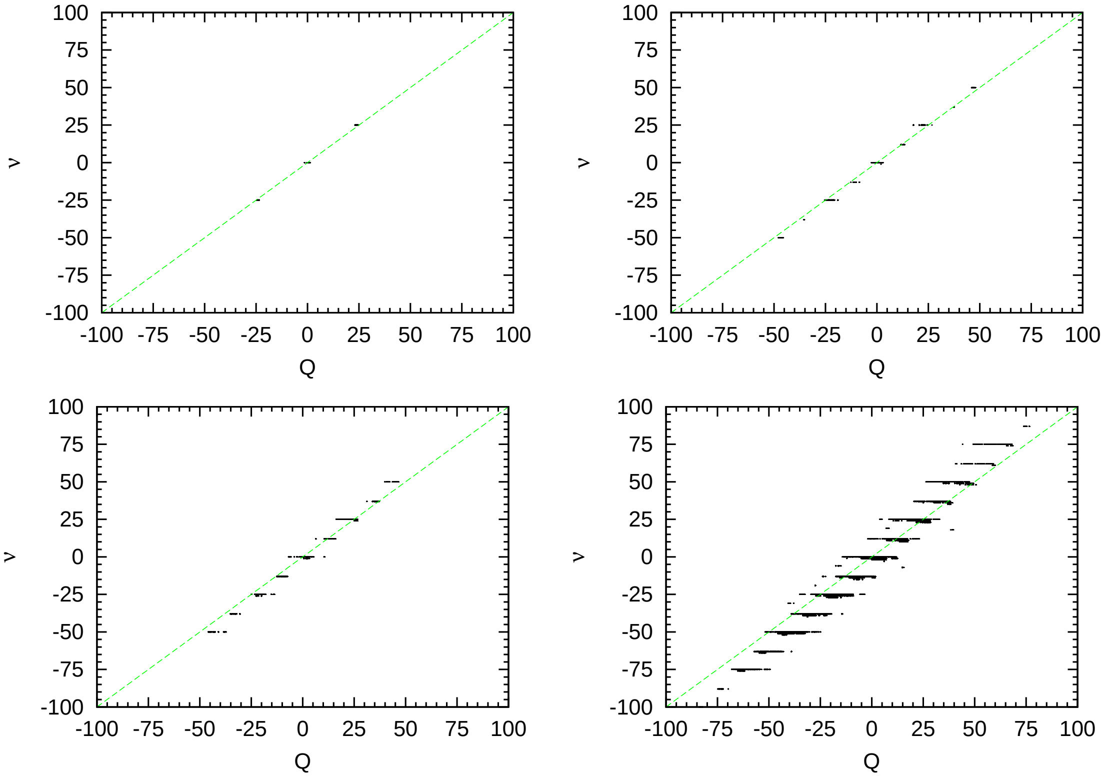

Figure 2: A scatter plot of the topological charge Q (x-axis) and the index ν (y-axis) for solutions at $N = 25$ with the action in the range $S/\beta \leq 100$ (top left), $100 \leq S/\beta \leq 200$ (top right) $200 \leq S/\beta \leq 300$ (bottom left) and $300 \leq S/\beta \leq 600$ (bottom right).

solutions in the displayed range without any restrictions.[6] We observe the accumulation of solutions with the topological charge close to a multiple integer of N. The region of action, for which we obtain only solutions with the topological charge close to a multiple integer of N, extends with N. This agrees with the argument that led to (3.18) in the previous section.

The minimum action in each topological sector is achieved by the $k = 1$ case, for which eqs. (3.16) and (3.17) lead to

$$S \simeq 4\pi^2\beta \left(\frac{Q}{N}\right)^2 \tag{4.1}$$

at large N. Note, however, that there are many solutions with $k > 1$ which have an action very close to (4.1). For solutions with larger action, on the other hand, the topological charge takes quite arbitrary values as expected.

We also observe the accumulation of solutions with the topological charge close to half-integer multiples of N. The minimum action achieved by such solutions increases linearly with N. These are the solutions having one 1×1 block $\Gamma_\mu^{(1)} = 1$ with $n_1 = 1$ and $m_1 = 0$.

[6]For $N = 75$ this calculation was quite time-consuming because there are so many classical solutions. However, the figure looks almost the same even if we restrict the number of blocks k in eq. (3.12) to be e.g., $k \leq 10$.

By choosing the m_j ($j \geq 2$) so that the arguments of the sine for the other blocks vanish in the large N limit, the topological charge (3.17) becomes

$$Q \simeq N \left(\sum_j m_j - M \right) + M + \frac{1}{2}, \tag{4.2}$$

which coincides with the observed spectrum noting that $M = \frac{N+1}{2}$. The action (3.16) is given by $S/\beta \simeq 4N$, which nicely explains our observation from figure 1.

Next let us calculate the index of the overlap Dirac operator for the classical solutions, and examine whether it agrees with the topological charge. In figure 2 we present a scatter plot of the topological charge Q (x-axis) and the index ν (y-axis) for solutions at $N = 25$ restricting the action in four different regions. We plot all the solutions in the displayed range without any restrictions. For $S/\beta \leq 100$, the index is either $\nu = 0$ or $\nu = \pm N$, and the topological charge turns out to be quite close to ν, which nicely confirms the index theorem. For solutions with larger action, we observe the case with ν close to half-integer multiples of N in accord with (4.2). While the index theorem is violated to some extent, there still exists a strong correlation between Q and ν. It is interesting that the smearing of the pattern occurs mainly in the direction of Q. In this regard let us recall that the definition (3.6) of Q we have used is just a naive descritized version of the continuum formula. In order to recover an exact index theorem in the discretized setting, one may have to use a more sophisticated definition as in the commutative case [47]. Whether this is possible or not is an interesting open question.

5. Configurations with the index of order 1

In the previous section we observed that the topological charge (3.17) and the index take only multiple integers of N for classical solutions with small action. This is in striking contrast to the corresponding commutative theory, where they take arbitrary integers, as we will discuss in section 6. In order to clarify the situation, let us construct configurations with the index of order 1 by interpolating the classical solutions in different topological sectors. This can be achieved by replacing the integer parameters m_j in the explicit form of the classical solutions (3.15) by real parameters. As the simplest case, we consider the solutions (3.12) with $k = 1$, and generalize them to a one-parameter family of configurations as

$$V_1 = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i\mu/N} & & & \\ & & e^{4\pi i\mu/N} & & \\ & & & \ddots & \\ & & & & e^{2\pi i(N-1)\mu/N} \end{pmatrix}, \tag{5.1}$$

where μ is a real parameter. Since $\mu = M$ gives the absolute minimum of the action, it is convenient to define $x \stackrel{\text{def}}{=} \mu - M$. As a function of x, the action and the topological charge

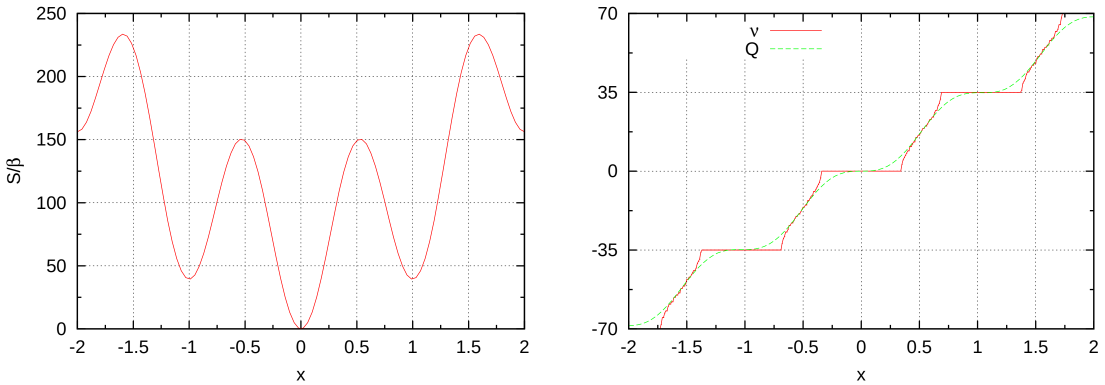

Figure 3: (Left) The action (5.2) is plotted as a function of x for $N = 35$. (Right) The topological charge Q in (5.3) and the index ν of the overlap Dirac operator are plotted as a function of x for $N = 35$.

can be evaluated as

$$S(x) = 4N\beta \left[(N-1)\sin^2 \frac{\pi x}{N} + \sin^2 \left\{ \pi \left(-1 + \frac{1}{N} \right) x \right\} \right] , \tag{5.2}$$

$$Q(x) = \frac{N}{2\pi} \left[(N-1)\sin \frac{2\pi x}{N} + \sin \left\{ 2\pi \left(-1 + \frac{1}{N} \right) x \right\} \right] . \tag{5.3}$$

In figure 3 (left) we plot the action $S(x)$ against x for $N = 35$. Let n be the integer which is closest to x, and consider the case where $|x - n| \sim O(N^{-p})$ with $p \geq 0$. Then, at large N, the leading contribution is given by

$$S(x)/\beta \simeq \begin{cases} 4\pi^2 n^2 & \sim \mathrm{O}(1) \qquad \text{for } p > \frac{1}{2} \\ 4N\sin^2(\pi x) & \sim \mathrm{O}(N^{1-2p}) \quad \text{for } 0 \leq p < \frac{1}{2} \ . \end{cases} \tag{5.4}$$

In figure 3 (right) we plot the index ν of the overlap Dirac operator and the topological charge $Q(x)$ against x for $N = 35$. As we increase x from 0 to 1, the index ν takes various integer values between 0 and N. In this way we are able to construct explicit configurations with ν of order 1. We have also studied $N = 15, 25$, and find that the result of the index ν is quite stable. For instance, the region of x which gives $\nu = 0$ is $|x| < 0.36$ for $N = 15$, and $|x| < 0.34$ for $N = 25, 35$. This implies, in view of (5.4), that the configurations with the index of order 1 constructed above has an action of order N.

We also observe in figure 3 (right) that the topological charge does not agree with the index for arbitrary x. When x is small, we obtain $Q \sim \frac{2}{3}N\pi^2 x^3$. Therefore, in order to obtain Q of order 1, we need to have $x \sim \mathrm{O}(N^{-1/3})$, for which the action becomes of order $N^{1/3}$ due to eq. (5.4). However, for such a small x, the index is zero in the large N limit according to our discussion in the previous paragraph. This shows that the index theorem is violated even in the large N limit if the action is as large as $\mathrm{O}(N^{1/3})$.[7] It is of course possible that the upper bound on S/β for which the index theorem holds *in general*

[7] We note that the admissibility condition derived in ref. [61] allows configurations with an action of $\mathrm{O}(N^2)$.

is actually less than $O(N^{1/3})$, say $O(1)$. In fact Q and ν agree when x is close to a half integer, for which S/β becomes of order N. We consider this as accidental, however, given the discrepancies observed for configurations with smaller action.

Incidentally, we note that the configurations at $x \sim n + \frac{1}{2}$ ($n \in \mathbb{Z}$), which gives the local maxima of the action $S(x)$, are closely related to the classical solutions with the topological charge close to half-integer multiples of N discussed in the previous section. Indeed, for the two types of configurations, the topological charge as well as the action coincides[8] at large N. In view of this, it is very likely that the classical solutions with the topological charge close to half-integer multiples of N actually correspond to saddle-point configurations, instead of being local minima of the action. These solutions are reminiscent of the sphaleron configurations [72].

6. Relationship to the commutative case

In this section we review some known results in the commutative case and discuss their relationship to our results. The commutative counterpart of our theory can be obtained from (2.7) by replacing the star-product with the ordinary product. The classical solutions are given by configurations with a uniform field strength. Explicitly, such a configuration can be constructed as

$$\begin{aligned} U_1(x) &= \begin{cases} 1 & \text{if } x_1 \neq a(N-1) \\ \exp(-2\pi i x_2 \tilde{Q}/aN) & \text{if } x_1 = a(N-1)\,, \end{cases} \\ U_2(x) &= \exp(2\pi i x_1 \tilde{Q}/aN^2)\,, \end{aligned} \tag{6.1}$$

where $\tilde{Q}$ is an integer, which corresponds to the topological charge.[9] In fact one may obtain other solutions by $U_\mu(x) \to e^{2\pi i h_\mu/N} U_\mu(x)$. Since configurations obtained in this way with h_μ differing by integers are related with each other by a large gauge transformation, the gauge inequivalent solutions are obtained by restricting h_μ within the range $-\frac{1}{2} \leq h_\mu < \frac{1}{2}$. Up to this degeneracy, which corresponds to the moduli space, there is essentially one classical solution in each topological sector labeled by an integer $\tilde{Q}$.

For $x_1 = a(N-1)$, $U_1(x)$ goes around the unit circle in the complex plane $\tilde{Q}$ times when x_2 goes from 0 to $a(N-1)$. This implies that the configuration is singular[10] in the continuum limit with finite $\tilde{Q}$ since $U_1(x) = 1$ for $x_1 \neq a(N-1)$. (Note, however, that the configuration is *physically* smooth since the field strength is constant.) This singularity disappears if and only if $\tilde{Q}$ is a multiple integer of N. In that case, the configuration itself

[8] Note, however, that they are not exactly equivalent configurations, since the eigenvalue spectrum of V_μ is different, and the former type of configuration is actually not a classical solution.

[9] A definition of the topological charge in the commutative case can be obtained from (3.6) by simply replacing the star-product by the ordinary product. This definition gives a non-integer value at finite lattice spacing, and approaches the correct integer value $\tilde{Q}$ only in the continuum limit. However, there is a simple geometric construction of the topological charge, which gives an integer value even at finite lattice spacing. This definition is used, for instance, in ref. [67].

[10] Note also that $U_2(x) \sim \exp(2\pi i \tilde{Q}/N)$ for $x_1 = a(N-1)$, while $U_2(x) = 1$ for $x_1 = 0$. Therefore, $U_2(x)$ becomes singular when $0 < \frac{\tilde{Q}}{N} < 1$ in the continuum limit. This singularity disappears, however, when $\tilde{Q}$ is kept finite in the continuum limit or when $\tilde{Q}$ is a multiple integer of N.

becomes totally smooth, and moreover it becomes translationally invariant in the direction 2. For such configurations, the star-product reduces to the ordinary product due to the definition (2.3). Therefore, the configurations satisfy the star-unitarity condition (2.8), which implies that they can be thought of as configurations on the discretized 2d NC torus. In fact one can easily show that they correspond to the classical solutions (3.12) given by a single block ($k = 1$).[11] Note, however, that there are many other classical solutions with larger action in each topological sector on the discretized 2d NC torus.

In the commutative case, the probability distribution of the topological sectors, which are labeled by the index ν, can be calculated exactly in the continuum,[12] and it turns out to be $P(\nu) \propto e^{-S(\nu)}$, where $S(\nu)$ is the minimum action in the topological sector ν. In terms of the lattice parameters, $S(\nu)$ may be written as[13]

$$S(\nu) = \frac{4\pi^2\beta}{N^2}\nu^2 \tag{6.2}$$

at large N and β. Since $\beta \propto \frac{1}{a^2}$ in the continuum limit, the distribution scales as a function of ν/ℓ, where $\ell = Na$ is the physical extent of the space. Note that the probability for obtaining $\nu \lesssim \mathrm{O}(\ell)$ remains finite.

In the NC case, the classical solutions with the action which is less than of order N exist only in the topological sectors labeled by ν which is a multiple of N. The minimum action (4.1) for classical solutions in these topological sectors agrees with (6.2) for the reason explained above. In the continuum limit, however, one has to take the $a \to 0$ limit in such a way that ϑ given by (3.5) is fixed. Since β should be sent to infinity as $\beta \propto \frac{1}{a^2} \propto N$, which follows from the scaling behavior of the correlation functions [18], we obtain finite action only for $\nu = 0$. This suggests that the probability of obtaining non-zero ν vanishes in the continuum limit, which is consistent with the instanton calculus in the continuum theory [59]. There the partition function has been written as a sum over all the instanton configurations with the total topological charge constrained to be equal to the magnetic flux, which is zero in the present case.

7. Summary and discussions

In this paper we have studied the index of the overlap Dirac operator in finite NC geometry, and clarified its basic properties including the index theorem. Our results confirm that the overlap Dirac operator indeed captures the topological nature of gauge theory in finite NC geometry, as in commutative lattice gauge theories. An analytic proof of the index theorem extending the works [50] in the commutative case would be an interesting future direction.

In fact we have observed a remarkable impact of NC geometry on the topological properties of the theory. As is well known, we encounter novel topological objects, which are represented by infinitely many classical solutions in each topological sector. However,

[11]These configurations have been studied earlier in refs. [68, 69] in the context of gauge theory in commutative space-time.

[12]We thank Hidenori Fukaya for clarification on this point.

[13]For studies of the probability distribution $P(\nu)$ on the lattice, see refs. [67, 70].

we also observe the opposite effects. The classical solutions with an action less than of order N should have an index ν which is a multiple integer of N. While we were able to construct configurations with the index ν of order 1 explicitly by interpolating the classical solutions, they have an action of order N. The classical solutions with $\nu = \pm N, \pm 2N, \cdots$ have an action of order 1, but since it is strictly positive and proportional to β, the action becomes infinite when one takes the $\beta \to \infty$ limit. Thus we are left with the $\nu = 0$ sector in the continuum limit.[14] Confirmation of this statement in the full quantum theory based on Monte Carlo simulation is reported in a separate paper [73].

The model we studied is the U(1) gauge theory on a discretized 2d NC torus, whose commutative counterpart has been studied extensively in the literature for the reason that it shares many dynamical properties with 4d non-abelian gauge theories. The conclusion that the path integral is dominated by the topologically trivial sector implies that the θ-term[15] is irrelevant unlike in the commutative case [71]. It would be interesting to investigate whether the suppression of non-zero indices is a general feature of gauge theories on NC geometry, which is independent of the space-time dimensionality, the gauge group, the matter content and so on. If the same property holds for the NC version of the standard model, it suggests an exciting possibility that the strong CP problem is naturally solved due to the effects of NC geometry.

Note, however, that 4d gauge theories in NC geometry has problems of its own. Unlike the 2d case studied here, the perturbative vacuum in the 4d case actually has tachyonic instability due to the UV/IR mixing [74−79]. The system stabilizes by "tachyon condensation", and finds a stable nonperturbative vacuum [19], in which the Wilson line corresponding to the tachyonic mode acquires a vacuum expectation value. Alternatively, one can stabilize the perturbative vacuum by introducing an appropriate UV cutoff. Although we do not know precisely how we should construct a realistic model at this moment, it is tempting to speculate that the strong CP problem may somehow be related to the physics of string theory origin.

Acknowledgments

It is our pleasure to thank Hidenori Fukaya, Satoshi Iso, Hikaru Kawai, Yusuke Kimura, Takeshi Morita, Kenji Ogawa and Kentaroh Yoshida for valuable discussions.

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[14] Repeating our analysis in the case of finite torus would be straightforward, but we consider that the topologically nontrivial configurations would be even more difficult to survive the continuum limit.

[15] The θ parameter, which appears here, represents the coefficient of the instanton number, and it should not be confused with the ϑ representing the noncommutativity of the space-time.

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